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On Harmonic Functions.[*]

By R. A. Harris.

§ 1.

This paper constitutes an elementary study in harmonic functions, V, and what might be called quasi-harmonic functions, V, where

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \ldots\ldots = 0, \text{ or } \nabla^2 V_p = 0, \tag{1}$$

and

$$\frac{\partial^2 \mathrm{V}}{\partial x^2} - \left(\frac{\partial^2 \mathrm{V}}{\partial y^2} + \ldots\ldots\right) = 0, \text{ or } \nabla^2 \mathrm{V}_p = 0, \tag{2}$$

p denoting the number of independent variables. While its object is to show that many solutions of (1) and (2) can be found by elementary methods and to point out some of the connections which exist between various solutions, the fact that many, and perhaps all, of the solutions may, in a sense, be regarded as an outgrowth or extension of those harmonic functions, X, Y, with which the theory of functions has to deal, will not be lost sight of.

The partial differential equations (1) and (2) will not be transformed into their equivalents in spherical or cylindrical coördinates even when solutions involving Legendre's coefficients, Bessel's functions, etc., are contemplated. In adapting equation (2) or any of its solutions to physical problems, the quantities x, y, z are generally to be replaced by $\varkappa t$, x, and y, respectively.

§ 2.

The lemmas or propositions given below are, with few exceptions, self-evident or nearly so. But inasmuch as their wording may not always be exact and such as to cause their meaning to be readily apprehended, brief demonstrations and qualifying statements are added.

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[*] An abstract of the principal parts of this paper was read before Section A of the American Association for the Advancement of Science, 1898. See *Bull. Am. Math. Soc.*, Vol. V (1898), pp. 96–98.

(i) If a solution of any linear partial differential equation having constant coefficients, but not a term of order zero, be differentiated (or integrated) with respect to one or several of its variables, the result is a solution of the given equation.

For, when several operators like $\frac{\partial}{\partial x}, \frac{\partial^2}{\partial x \partial y}, \ldots.$ are applied to any quantity or solution the order of application is immaterial; the result of any number of such operations on the given differential equation is, of course, a new differential equation. So if a partial derivative of the function constituting the given solution be made, by transposing the operators, to occupy the same position in the new differential equation as did the given solution in the original equation, the result must still be a true equation.

Similarly for partial integration; but it is understood that in this case the number of terms in the power-series representing the given solution developed in powers of the variable in question, is generally infinite.

(ii) If in any solution of an equation of the kind just referred to, any two of the variables be replaced by their equivalents in circular coördinates, the result of differentiating (or integrating) the solution with respect to either circular coördinate is a solution of the given equation.

For, suppose $\eta = y + \sqrt{-1}\,z, \quad \zeta = y - \sqrt{-1}\,z,$ and so

$$y = \tfrac{1}{2}(\eta + \zeta), \quad \sqrt{-1}\,z = \tfrac{1}{2}(\eta - \zeta).$$

$$\frac{\partial}{\partial \eta} = \frac{\partial}{\partial y}\frac{dy}{d\eta} + \frac{\partial}{\partial z}\frac{dz}{d\eta} = \tfrac{1}{2}\left(\frac{\partial}{\partial y} - \sqrt{-1}\,\frac{\partial}{\partial z}\right),$$

$$\frac{\partial}{\partial \zeta} = \tfrac{1}{2}\left(\frac{\partial}{\partial y} + \sqrt{-1}\,\frac{\partial}{\partial z}\right),$$

and so, by lemma (i), these indicated differentiations yield solutions. Similarly for coördinates such that, say, $y + z = \eta_{,}$, $y - z = \zeta_{,}$, which may be called hyperbolic coördinates.

The same is evidently true if any other linear substitutions be made for the variables.

(iii) If in any solution of (1) the letters $x, y, z, \ldots.$ be interchanged at will, the result is a solution. The same is true for all variables after the first in a solution of (2).

For, the operators in (1) are symmetrical in all variables, and in (2), in all but the first variable.

(iv) If in any solution, V_p, of (1), $y, z, \ldots.$ be replaced by $\pm\sqrt{-1}\,y$, $\pm\sqrt{-1}\,z, \ldots.$, the result is a solution, V_p, of (2); and conversely.

For, equation (1) must still be a true equation provided the substitution be made in the operators as well as in the original function V_p. But the operators $\frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}, \ldots.$ become $-\frac{\partial^2}{\partial y^2}, -\frac{\partial^2}{\partial z^2}, \ldots.$, and so the transformed function must satisfy (2).

(v) Given any solution of (1) or (2) in the form

$$f_0(y, z, \ldots.)\,\chi(x) + f_2(y, z, \ldots.)\,\chi''(x) + f_4(y, z, \ldots.)\,\chi^{\mathrm{IV}}(x) + \ldots., \tag{3}$$

and such that the entire coefficients of $\chi''(x)$, $\chi^{\mathrm{IV}}(x), \ldots.$ in (1) or (2) are severally zero, then is

$$f_0(y, z, \ldots.)\,\psi(x) + f_2(y, z, \ldots.)\,\psi''(x) + f_4(y, z, \ldots.)\,\psi^{\mathrm{IV}}(x) + \ldots. \tag{4}$$

a solution, ψ being an arbitrary function.

This means that the given solution when substituted in (1) or (2) satisfies the same irrespective of the nature of χ, and so χ may be replaced by ψ. To say that the coefficient of $\chi^{(\nu)}$ in (1) or (2) shall be zero, is equivalent to saying that

$$f_{\nu-2}(y, z, \ldots.) \pm \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \ldots.\right) f_\nu(y, z, \ldots.) = 0. \tag{5}$$

Of course this is likewise true for the alternate terms; that is, for the terms having odd subscripts and indices.

(vi) If in any solution of (1) or (2) referred to in lemma (v), $x, y, \ldots.$ be replaced by $x \pm x'$, $y \pm y', \ldots.$, where $x', y', \ldots.$ are constants, the result is also a solution of the given differential equation.

For, let $x_{\prime} = x \pm x'$, $y_{\prime} = y \pm y'$, $\ldots.$; the solution will be the same function of $x_{\prime}, y_{\prime}, \ldots.$ as it originally was of $x, y, \ldots.$. The operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \ldots.$ may be written $\frac{\partial}{\partial x_{\prime}}, \frac{\partial}{\partial y_{\prime}}, \ldots.$. But expression (5) being identically zero, it will remain so when $x_{\prime}, y_{\prime}, \ldots.$ replace $x, y, \ldots.$ throughout.

(vii) If for the variables in equation (1) we make an orthogonal substitution, the result is still a true equation.

For, in V, replacing $x, y, \ldots.$ by their equivalents in a second set of variables $x_{\prime}$, $y_{\prime}$, ..., the form of V will generally be altered, but its value will, of course, remain the same as before. Consequently the operator

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\cdots\cdot$$

applied to this new expression will still give a result equal to zero. But the effect of the operator

$$\frac{\partial^2}{\partial x_{\prime}^2}+\frac{\partial^2}{\partial y_{\prime}^2}+\cdots\cdot$$

upon either form of V is not zero unless the two operators are equal or proportional. The first operator when expressed in terms of $x_{\prime}$, $y_{\prime}$, ... by aid of the matrix of transformation shows that for any orthogonal substitution the two operators are equivalent, ignoring a constant factor, the modulus of the substitution. For suppose

$$\left.\begin{aligned} x &= a_1 x_{\prime} + b_1 y_{\prime} + \cdots\cdot,\\ y &= a_2 x_{\prime} + b_2 y_{\prime} + \cdots\cdot,\\ &\cdots\cdots\cdots\cdots\cdots, \end{aligned}\right\} \text{ and so } \left\{\begin{aligned} x_{\prime} &= a_1 x + a_2 y + \cdots\cdot,\\ y_{\prime} &= b_1 x + b_2 y + \cdots\cdot,\\ &\cdots\cdots\cdots\cdots\cdots; \end{aligned}\right.$$

then

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial x_{\prime}}\frac{dx_{\prime}}{dx}+\frac{\partial}{\partial y_{\prime}}\frac{dy_{\prime}}{dx}+\cdots\cdot=a_1\frac{\partial}{\partial x_{\prime}}+b_1\frac{\partial}{\partial y_{\prime}}+\cdots\cdot,$$

and finally,

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\cdots\cdot=(a_1^2+a_2^2+\cdots\cdot)\frac{\partial^2}{\partial x_{\prime}^2}+(b_1^2+b_2^2+\cdots\cdot)\frac{\partial^2}{\partial y_{\prime}^2}+\cdots\cdot,$$

wherein

$$a_1^2+a_2^2+\cdots\cdot=b_1^2+b_2^2+\cdots\cdot=\cdots\cdot\cdot$$

Similarly, if only a part or group of the variables of (1) be transformed into a group of corresponding variables by means of a uni-modular orthogonal substitution, the equation will still hold true. This remark is evidently applicable to the second part of equation (2).

(viii) Equation (1) is satisfied by making V equal to

$$\frac{d^{p-2}\log\rho}{d\rho^{p-2}} \text{ or } \frac{1}{\rho^{p-2}}, \tag{6}$$

where p denotes the number of variables in V and where

$$\rho\equiv\sqrt{x^2+y^2+\cdots\cdot\cdot}$$

For, substituting $1/\rho^{p-2}$ in (1) we have the sum of p quantities like

$$(p-2)(-\rho^{-p}+p\,x^2\rho^{-p-2}),$$

except that x^2 is replaced by y^2, z^2, ... in succession; and this sum is zero.

Similarly, or by lemma (iii),

$$\frac{1}{\sigma^{p-2}}, \tag{7}$$

where $\qquad \sigma \equiv \sqrt{x^2-(y^2+\ldots\ldots)},$

is a solution of equation (2).

(ix) If H_n denote an homogeneous solution of (1) of degree n, then is

$$\frac{H_n}{\rho^{2n+p-2}} \tag{8}$$

likewise a solution, n being general.

For, operating upon H_n/ρ^{2n+p-2} with $\dfrac{\partial^2}{\partial x^2}$, we have

$$\frac{1}{\rho^{2n+p-2}}\frac{\partial^2 H_n}{\partial x^2}-\frac{2x(2n+p-2)}{\rho^{2n+p}}\frac{\partial H_n}{\partial x}-\frac{2n+p+2}{\rho^{2n+p}}H_n + \frac{x^2(2n+p-2)(2n+p)}{\rho^{2n+p+2}}H_n.$$

Replacing x by y, we have the result of the operation $\dfrac{\partial^2}{\partial y^2}$; and so on. But H_n being a solution, the result of adding together the terms involving $\dfrac{\partial^2 H_n}{\partial x^2}$, $\dfrac{\partial^2 H_n}{\partial y^2}$, ... must be zero. Because H_n is homogeneous and of degree n, the sum of the second terms is $-\dfrac{2n(2n+p-2)}{\rho^{2n+p}}H_n$. The sum of the third terms is $-\dfrac{p(2n+p+2)}{\rho^{2n+p}}H_n$; the sum of the fourth terms, since $\rho^2=x^2+y^2+\ldots\ldots$, is $\dfrac{(2n+p-2)(2n+p)}{\rho^{2n+p}}H_n$. Hence the total is zero.

Conversely, if (8) be a solution and H_n an homogeneous function, H_n must be a solution.

Similarly, or by lemma (iii), if H_n denote an homogeneous solution of (2),

$$\frac{H_n}{\sigma^{2n+p-2}}, \tag{9}$$

where

$$\sigma=\sqrt{x^2-(y^2+z^2+\ldots\ldots)},$$

is a solution.

§ 3. *Application of the Foregoing Lemmas to Spherical Harmonic Functions.*

If the solutions (6), (7) be differentiated n times with respect to x, we obtain solutions of the forms

$$\frac{F_n\left(\frac{x}{\rho}\right)}{\rho^{n+p-2}}, =\frac{\rho^n F_n\left(\frac{x}{\rho}\right)}{\rho^{2n+p-2}}; \quad \frac{F_n\left(\frac{x}{\sigma}\right)}{\sigma^{n+p-2}}, =\frac{\sigma^n F_n\left(\frac{x}{\sigma}\right)}{\sigma^{2n+p-2}}, \tag{10}$$

where $F_n\left(\frac{x}{\rho}\right)$, $F_n\left(\frac{x}{\sigma}\right)$ denote algebraic integral functions of the n-th degree in x/ρ, x/σ.

The truth of this statement becomes evident upon performing a few differentiations and referring to lemma (i). The converse of (ix) shows that the homogeneous functions

$$\rho^n F_n\left(\frac{x}{\rho}\right), \quad \sigma^n F_n\left(\frac{x}{\sigma}\right) \tag{11}$$

are also solutions of equations (1) and (2). These solutions evidently have circular symmetry about the x-axis; they are termed zonal harmonics of the n-th order while (10) are zonal harmonics of order $-n-p+2$.

If the solution $1/\rho^{p-2}$ be differentiated with respect to a circular coördinate, ζ, we have

$$\frac{\partial^n \frac{1}{\rho^{p-2}}}{\partial \zeta^n}=(-1)^n\frac{(p-2)p(p+2)\ldots\ldots(p+n-1)}{2^n}\frac{\eta^n}{\rho^{p+2(n-1)}}, \tag{12}$$

where η and ζ are conjugate imaginaries.

Similarly, if the solution of (2), $1/\sigma^{p-2}$, be thus differentiated, we have

$$\frac{\partial^n \frac{1}{\sigma^{p-2}}}{\partial \zeta^n}=(+1)^n\frac{(p-2)p(p+2)\ldots\ldots(p+n-1)}{2^n}\frac{\eta^n}{\sigma^{p+2(n-1)}}. \tag{13}$$

If $\eta_{\prime}$, $\zeta_{\prime}$ denote hyperbolic coördinates such that

$$\eta_{\prime}=x+y\,(\text{or } z \text{ or } w, \ldots\ldots),$$
$$\zeta_{\prime}=x-y\,(\text{or } z \text{ or } w, \ldots\ldots),$$

the n-th derivative with respect to $\zeta_{\prime}$ is

$$\frac{\partial^n \frac{1}{\sigma^{p-2}}}{\partial \zeta_{\prime}^n}=(-1)^n\frac{(p-2)p(p+2)\ldots\ldots(p+n-1)}{2^n}\frac{\eta_{\prime}^n}{\sigma^{p+2(n-1)}}. \tag{14}$$

In all of the above cases η and ζ (or $\eta_{,}$ and $\zeta_{,}$) can be interchanged at will. By (ii) and (vii), the expressions involving ρ are solutions of (1), and those involving σ, solutions of (2).

If the solutions

$$\frac{1}{\rho^{p-2}} \text{ and } \frac{1}{\sigma^{p-2}}, \text{ or } \frac{1}{(x^2 \pm c_{,}^2)^{\frac{p-2}{2}}},$$

where $c_{,}^2 = y^2 + z^2 + \ldots.$, be integrated one or more times with respect to x, we obtain solutions of (1) and (2).

Making $p = 3$ and performing one integration, we obtain from the solutions $\pm \frac{1}{r}$, $\pm \frac{1}{\sigma}$, the solutions

$$\frac{1}{2} \log \frac{r + x}{r - x}, \quad \frac{1}{2} \log \frac{\sigma + x}{\sigma - x}. \tag{15}$$

The former may be written

$$\frac{1}{2} \log \frac{1 + \mu}{1 - \mu} = r^0 Q_0(\mu), \tag{16}$$

where $\mu = x/r$. Another integration gives the solutions

$$\frac{1}{2} x \log \frac{r + x}{r - x} - r, \quad \frac{1}{2} x \log \frac{\sigma + x}{\sigma - x} - \sigma. \tag{17}$$

The former may be written

$$r\left(\frac{1}{2} \mu \log \frac{1 + \mu}{1 - \mu} - 1\right) = r\, Q_1(\mu). \tag{18}$$

Solutions like these are zonal harmonics of the second kind.

Zonal Harmonics, $p = 3$. By lemma (viii) or by direct substitution, we see that

$$V_3 = \frac{1}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{x} - \frac{1}{2}\frac{y^2 + z^2}{x^3} + \frac{1.3}{2.4}\frac{(y^2 + z^2)^2}{x^5} - \frac{1.3.5}{2.4.6}\frac{(y^2 + z^2)^3}{x^7} + \ldots. \tag{19}$$

satisfies equation (1). Moreover it is a solution of the form referred to in (v), $\chi(x)$ being $1/x$. Therefore, by (vi),

$$\frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \tag{20}$$

satisfies equation (1), as can be directly verified. The first radical denotes the distance of a variable point from the origin; the second, the distance of this point from a fixed point whose coördinates are x', y', z', both points being referred to the same coördinate axes. Suppose the fixed point to lie upon the x- (or x'-) axis; *i. e.*, suppose $y' = z' = 0$. The distance of x, y, z from x', 0, 0 is $\sqrt{(x-x')^2 + y^2 + z^2}$ or $\sqrt{r^2 - 2xr' + r'^2}$, where $r^2 = x^2 + y^2 + z^2$, $r'^2 = x'^2 + 0^2 + 0^2 = x'^2$. Expanding by the binomial theorem and arranging the result in powers of r/r' or r'/r, we have

$$\frac{1}{\sqrt{r^2 - 2xr' + r'^2}} = \frac{1}{r'\sqrt{1 - 2\dfrac{x}{r}\dfrac{r}{r'} + \dfrac{r^2}{r'^2}}} = \frac{1}{r\sqrt{1 - 2\dfrac{x}{r}\dfrac{r'}{r} + \dfrac{r'^2}{r^2}}} \tag{21}$$

$$= \frac{1}{r'}\left[P_0 + P_1\frac{r}{r'} + P_2\frac{r^2}{r'^2} + \dots\right] \tag{22}$$

$$= \frac{1}{r}\left[P_0 + P_1\frac{r'}{r} + P_2\frac{r'^2}{r^2} + \dots\right]. \tag{23}$$

In these developments it is evident that P denotes a rational integral algebraic function of x/r. But (22) or (23) satisfies Laplace's equation (1), regardless of the value assigned to the constant r' or to the ratio r'/r. Consequently since (22) and (23) are arranged in powers of this arbitrary magnitude, each of their terms is likewise a solution. In other words,

$$r^n P_n\left(\frac{x}{r}\right) \text{ or } r^{-n-1} P_n\left(\frac{x}{r}\right), \tag{24}$$

where $n = 0, 1, 2, \dots$, is a solution. [Cf. lemma (ix).]

We may write $r^n P_n\left(\frac{x}{r}\right)$ in the form

$$A_n x^n + B_n x^{n-2} r^2 + C_n x^{n-4} r^4 + \dots, \tag{25}$$

or

$$K_{n,0}\, x^n + K_{n,2}\, x^{n-2} r^2 + K_{n,4}\, x^{n-4} r^4 + \dots, \tag{26}$$

or

$$\sum_{\nu = 0, 2, 4, \dots} K_{n,\nu}\, x^{n-\nu} r^{\nu}. \tag{27}$$

Expressions for A_n, B_n, C_n, ... may be obtained from the expansion of (21) in powers of r/r', by picking out the entire coefficient of the n-th power. The sum of the coefficients entering this entire coefficient may be shown to be unity.

Laplace's Coefficients. Returning to solution (20), in which x', y', z' may refer to any fixed point in space, we have, since $r'^2 = x'^2 + y'^2 + z'^2$,

$$\left.\begin{aligned} &\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \\ &= \frac{1}{r'\sqrt{1 - 2\dfrac{xx' + yy' + zz'}{rr'}\dfrac{r}{r'} + \dfrac{r^2}{r'^2}}} = \frac{1}{r\sqrt{1 - 2\dfrac{xx' + yy' + zz'}{rr'}\dfrac{r'}{r} + \dfrac{r'^2}{r^2}}}\cdot \end{aligned}\right\} \tag{28}$$

These expanded by the binomial theorem and arranged in powers of r/r' or r'/r may be written

$$\frac{1}{r'}\left[Y_0 + Y_1\frac{r}{r'} + Y_2\frac{r^2}{r'^2} + \dots\right], \tag{29}$$

$$\frac{1}{r}\left[Y_0 + Y_1\frac{r'}{r} + Y_2\frac{r'^2}{r^2} + \dots\right]. \tag{30}$$

It is evident that Y denotes the same rational integral function of $\dfrac{xx' + yy' + zz'}{rr'}$ that P (Legendre's coefficient) denotes of x/r. The Y's are Laplace's coefficients. The expression $\dfrac{xx' + yy' + zz'}{rr'}$ is the cosine of the angle at the origin between the point x, y, z and the point x', y', z'. Since the value of r'/r is arbitrary, each term of the above expressions is a solution of (1).

Tesseral Harmonics, Associated Forms. Any unimodular orthogonal substitution does not alter the value of r, and the result is by lemma (vii) a solution of equation (1).

In the solutions $r^n P_n$ and $r^{-n-1} P_n$, r and so r^n and r^{-n-1} are unaltered in value if we put $x = l_1 x_{,} + m_1 y_{,} + n_1 z_{,}$, where l_1, m_1, n_1 are quantities subject to no other condition than $l_1^2 + m_1^2 + n_1^2 = 1$. Therefore, if $l_1 = 1$, then $m_1^2 + n_1^2 = 0$ or $n_1 = \sqrt{-1}\, m_1$. Let $m_1 = \alpha_1$; then $n_1 = \sqrt{-1}\,\alpha_1$ and $x = x_{,} + \alpha_1(y_{,} + \sqrt{-1}\, z_{,})$. In substituting this value for x in expression (25), we have to consider only the explicit x since r remains unchanged. The result of the substitution arranged in powers of $\alpha_1(y_{,} + \sqrt{-1}\, z_{,})$ will be a series of expressions of the form

$$\left.\begin{aligned} \alpha_1^s(y + \sqrt{-1}\, z)^s\,[n(n-1)\dots(n-s+1)\,A_n x^{n-s} \\ + (n-2)(n-3)\dots(n-s-1)\,B_n x^{n-s-2} r^2 \\ + (n-4)(n-5)\dots(n-s-3)\,C_n x^{n-s-4} r^4 + \dots], \end{aligned}\right\} \tag{31}$$

wherein all subscripts have been omitted from the variables. Each expression of the above form is a solution of the equation (1); for, the entire expression (25) is still a solution after the above transformation has been made, whatever be the value of α_1. Consequently, α_1 being arbitrary, the entire coefficient of any power of α_1 must be a solution. Further, the real part will be one solution, the imaginary part another.

By writing $x = r\cos\theta$, $= r\mu$, $y = r\sin\theta\cos\phi$, $z = r\sin\theta\sin\phi$, expression (31) when divided by α_1^s becomes

$$r^n \sin^s\theta\,(\cos s\phi + \sqrt{-1}\sin s\phi)\frac{d^s P_n}{d\mu^s}, \tag{32}$$

which may be written

$$r^n(\cos s\phi + \sqrt{-1}\sin s\phi)\,T_n^{(s)}. \tag{33}$$

Expression (27) may have its $y^2 + z^2$ replaced by the equivalent, $\eta\zeta$, in circular coördinates and, by lemma (ii), the result of one or more differentiations with respect to ζ will be a solution.

$$\frac{\partial}{\partial \zeta} \sum_{\nu = 0, 2, 4, \ldots\ldots} K_{n,\nu}\, x^{n-\nu}\, r^{\nu} = \eta \sum_{\nu = 2, 4, \ldots\ldots} \frac{\nu}{2} K_{n,\nu}\, x^{n-\nu}\, r^{\nu-2}. \tag{34}$$

By making $s = 1$ in (31), the latter becomes, to a factor,

$$\eta \sum_{\nu = 0, 2, 4, \ldots\ldots} (n - \nu)\, K_{n,\nu}\, x^{n-1-\nu}\, r^{\nu}. \tag{35}$$

Upon replacing n by $n - 1$, ν by $\nu - 2$, this becomes

$$\eta \sum_{\nu = 2, 4, \ldots\ldots} (n - \nu + 1)\, K_{n-1,\, \nu-2}\, x^{n-\nu}\, r^{\nu-2}. \tag{36}$$

Here are two solutions (34), (36) of equation (1), both of which are identical so far as the variables are concerned. Consequently, the coefficients of like powers of the variables must have a constant ratio to each other.

$$\therefore \frac{\nu}{2\,(n - \nu + 1)} \frac{K_{n,\,\nu}}{K_{n-1,\,\nu-2}} = \text{constant}, = \frac{1}{2}, \tag{37}$$

as any special case will show. This furnishes a ready means of deriving P_n from P_{n-1}; for

$$K_{n,\,\nu} = -\frac{n - \nu + 1}{\nu} K_{n-1,\,\nu-2}, \tag{38}$$

$$= -\frac{(n-1) - (\nu - 2)}{\nu} K_{n-1,\,\nu-2}. \tag{39}$$

Example. Given $P_1 = \frac{x}{r}$ or μ, to determine P_2, P_3, P_4, ... Here $K_{1,0} = 1$;

$$\therefore K_{2,2} = -\frac{2 - 2 + 1}{2} K_{1,0} = -\frac{1}{2}.$$

The sum of all coefficients in a P function is to be taken equal to unity.

$$\therefore P_2 = \frac{3}{2}\frac{x^2}{r^2} - \frac{1}{2}.$$

Similarly,

$$P_3 = \frac{5}{2}\frac{x^3}{r^3} - \frac{3}{2}\frac{x}{r},$$

$$P_4 = \frac{35}{8}\frac{x^4}{r^4} - \frac{30}{8}\frac{x^2}{r^2} + \frac{3}{8}.$$

Other matters concerning spherical harmonics will be found in §§ 4 and 6.

§ 4. *Applications of Lemma* (v).

The solutions $1/\rho^{p-2}$, $1/\sigma^{p-2}$ of equations (1) and (2) become when expanded in powers of x and $c_{,}$, where $c_{,}^2 = y^2 + z^2 + \dots\dots$,

$$\sum_{\nu=0}^{\infty} (\mp 1)^{\nu} \frac{(p-2)p(p+2)\dots\dots(p+2\nu-4)}{2^{\nu}} \frac{1}{\nu!} \frac{c_{,}^{2\nu}}{x^{p+2\nu-2}}. \tag{40}$$

In particular, if $p = 3$,

$$\frac{1}{r} = \frac{1}{x} - \frac{1}{2}\frac{y^2+z^2}{x^3} + \frac{1.3}{2.4}\frac{(y^2+z^2)^2}{x^5} - \frac{1.3.5}{2.4.6}\frac{(y^2+z^2)^3}{x^7} + \dots\dots \tag{41}$$

If $\chi(x) \equiv \frac{1}{x}$, then $\chi''(x) = \frac{2!}{x^3}$, $\chi^{\text{IV}}(x) = \frac{4!}{x^5}$, $\dots\dots$;

$$\therefore \frac{1}{r} = \chi(x) - \frac{1}{2}\frac{1}{2!}(y^2+z^2)\chi''(x) + \frac{1.3}{2.4}\frac{1}{4!}(y^2+z^2)^2\chi^{\text{IV}}(x) - \dots\dots \tag{42}$$

Let $\psi(x) = \sinh x$; then $\psi''(x) = \sinh x$; $\psi^{\text{IV}}(x) = \sinh x$, $\dots\dots$;

$$\therefore \sinh x\left[1 - \frac{1}{2^2}(y^2+z^2) + \frac{1}{2^2\,4^2}(y^2+z^2)^2 - \dots\dots\right], \tag{43}$$

or

$$\sinh x\, J_0(\varpi), \tag{44}$$

where $\varpi^2 = y^2 + z^2$ and J_0 denotes Bessel's function of order zero, is a solution of Laplace's equation (1).

If $\psi(x) = x^n$, then $\psi''(x) = n(n-1)x^{n-2}$,

$$\psi^{\text{IV}}(x) = n(n-1)(n-2)(n-3)x^{n-4}, \dots\dots;$$

$$\therefore x^n\left[1 - \frac{n(n-1)}{2^2}\frac{y^2+z^2}{x^2} + \frac{n(n-1)(n-2)(n-3)}{2^2\,4^2}\frac{(y^2+z^2)^2}{x^4} - \dots\dots\right] \tag{45}$$

is also a solution. Putting $x = r\mu$, this may be written

$$r^n\mu^n\left[1 - \frac{n(n-1)}{2^2}\frac{1-\mu^2}{\mu^2} + \frac{n(n-1)(n-2)(n-3)}{2^2\,4^2}\frac{(1-\mu^2)^2}{\mu^4} - \dots\dots\right], \tag{46}$$

or

$$r^n P_n(\mu),$$

since, by writing $r^2 - x^2$ for $y^2 + z^2$, (45) is of the form (25) and $r^n P_n(\mu)$ satisfies equation (1).

[It may be well to note that the general expression (45) for a solid zonal harmonic can be obtained by starting with the solution $x^2 - \frac{1}{2}(y^2 + z^2)$, where $\chi(x) = x^2$, proceeding to one of the third degree by making $\psi(x) = x^3$, and then to a solution of the fourth degree by making $\psi(x) = x^4$, and so on.

The coefficient of the added term in the solution of the fourth degree, viz., $K_4(y^2+z^2)^2\psi^{\text{IV}}(x)$, is found by substituting the solution in equation (1). In general, it will be found, upon equating to zero the entire coefficient of $x^{n-\nu}y^{\nu-2}$ in (1), that $\nu^2K_\nu = -K_{\nu-2}$. Hence, any solid zonal harmonic may be written

$$x^n - \frac{y^2+z^2}{2^2}\frac{\partial^2(x^n)}{\partial x^2} + \frac{(y^2+z^2)^2}{2^2.\,4^2}\frac{\partial^4(x^n)}{\partial x^4} - \ldots.,$$

which agrees with (45).]

Replacing n by $-n-1$, we obtain the solution

$$\frac{1}{r^{n+1}\mu^{n+1}}\Bigg[1 - \frac{(n+1)(n+2)}{2^2}\frac{1-\mu^2}{\mu^2} + \frac{(n+1)(n+2)(n+3)(n+4)}{2^2\,4^2}\frac{(1-\mu^2)^2}{\mu^4} - \ldots.\Bigg], \tag{47}$$

which, by lemma (ix), must be equivalent to P_n/r^{n+1}.

Since $(y+\sqrt{-1}\,z)^n$ is a solution of equation (1) of the n-th degree and homogeneous, we have by lemma (ix) as another solution this expression divided by r^{2n+1},

$$\frac{(y+\sqrt{-1}\,z)^n}{r^{2n+1}} = \frac{(y+\sqrt{-1}\,z)^n}{x^{2n+1}}\Bigg[1 - \frac{2n+1}{2}\frac{y^2+z^2}{x^2} + \frac{2n+1}{2}\,\frac{2n+3}{2}\,\frac{1}{2!}\,\frac{(y^2+z^2)^2}{x^4} - \frac{2n+1}{2}\,\frac{2n+3}{2}\,\frac{2n+5}{2}\,\frac{1}{3!}\,\frac{(y^2+z^2)^3}{x^6} + \ldots.\Bigg], \tag{48}$$

which may, by putting $\chi(x) = 1/x^{2n+1}$, be written

$$(y+\sqrt{-1}\,z)^n\Bigg[\chi(x) - \frac{y^2+z^2}{2^2(n+1)}\chi''(x) + \frac{(y^2+z^2)^2}{2^4\,2!\,(n+1)(n+2)}\chi^{\text{IV}}(x) - \frac{(y^2+z^2)^3}{2^6\,3!\,(n+1)(n+2)(n+3)}\chi^{\text{VI}}(x) + \ldots.\Bigg]. \tag{49}$$

Putting $\psi(x) = \sinh x$, we have as a solution

$$\sinh x\,(y+\sqrt{-1}\,z)^n\Bigg[1 - \frac{y^2+z^2}{2^2(n+1)} + \frac{(y^2+z^2)^2}{2^4\,2!\,(n+1)(n+2)} - \frac{(y^2+z^2)^3}{2^6\,3!\,(n+1)(n+2)(n+3)} + \ldots.\Bigg]. \tag{50}$$

Since $(y+\sqrt{-1}\,z)^n = (\cos n\phi + \sqrt{-1}\,\sin n\phi)\,(y^2+z^2)^{n/2}$, this solution may be written

$$c_n \sinh x\,(\cos n\phi + \sqrt{-1}\,\sin n\phi)\,J_n(\varpi), \tag{51}$$

where c_n denotes a constant dependent upon n.

If $\psi(x)=x^{n-s}$, then (49) becomes

$$(y+\sqrt{-1}\,z)^s x^{n-s}\left[1-\frac{(n-s)(n-s-1)}{2^2(s+1)}\,\frac{y^2+z^2}{x^2}\right.$$
$$\left.+\frac{(n-s)(n-s-1)(n-s-2)(n-s-3)}{2^4\,2!\,(s+1)(s+2)}\,\frac{(y^2+z^2)^2}{x^4}-\dots\right], \tag{52}$$
$$=(y+\sqrt{-1}\,z)^s (r\mu)^{n-s}\left[1-\frac{(n-s)(n-s-1)}{2^2(s+1)}\,\frac{1-\mu^2}{\mu^2}\right.$$
$$\left.+\frac{(n-s)(n-s-1)(n-s-2)(n-s-3)}{2^4\,2!\,(s+1)(s+2)}\,\frac{(1-\mu^2)^2}{\mu^4}-\dots\right]. \tag{53}$$

Since $y^2+z^2=r^2-x^2$, (52) can be broken up into terms containing the same powers of the variables as do the terms of (31); therefore their coefficients are proportional and so (53) may be written

$$c_{n,s}\, r^n (\cos s\phi+\sqrt{-1}\,\sin s\phi)\, T_n^{(s)}, \tag{54}$$

where $c_{n,s}$ denotes a constant dependent upon n and s.

It is obvious that if in (46) divided by r^n, or

$$P_n(\cos\theta)=\cos^n\theta\left[1-\frac{n(n-1)}{2^2}\tan^2\theta+\frac{n(n-1)(n-2)(n-3)}{2^2\,4^2}\tan^4\theta-\dots\right], \tag{55}$$

we put $\theta=\varpi/n$, and then cause n to become infinite, we shall have

$$\mathop{\text{Limit}}_{n=\infty} P_n\left(\cos\frac{\varpi}{n}\right)=1-\frac{\varpi^2}{2^2}+\frac{\varpi^4}{2^2\,4^2}-\dots=J_0(\varpi). \tag{56}$$

Similarly, if in (53) or

$$r^{n-s}(y+\sqrt{-1}\,z)^s\cos^{n-s}\theta\left[1-\frac{(n-s)(n-s-1)}{2^2(s+1)}\tan^2\theta\right.$$
$$\left.+\frac{(n-s)(n-s-1)(n-s-2)(n-s-3)}{2^4\,2!\,(s+1)(s+2)}\tan^4\theta-\dots\right] \tag{57}$$

we put $\theta=\varpi/s$, and then cause s to become infinite in the part which is multiplied by $r^{n-s}(\cos\phi+\sqrt{-1}\,\sin\phi)^s$, we shall have for this part

$$\varpi^s\left[1-\frac{\varpi^2}{2^2(s+1)}+\frac{\varpi^4}{2^4\,2!\,(s+1)(s+2)}-\dots\right]=c'_s J_s(\varpi), \tag{58}$$

where c'_s denotes a constant dependent upon s.

According to lemma (ix), x/r^3 is a solution; it is, in fact, the velocity-potential defining the motion of a sphere in an infinite mass of water. x/r^3 is the kind of solution satisfying condition (5). If in

$$\frac{x}{r^3} = \frac{1}{x^2} - \frac{1\,.\,3}{2}\frac{y^2+z^2}{x^4} + \frac{1\,.\,3\,.\,5}{2\,.\,2}\frac{1}{2\,!}\frac{(y^2+z^2)^2}{x^6} - \ldots . \tag{59}$$

we put $\chi(x) = \frac{1}{x^2}$, then $\chi''(x) = \frac{2\,.\,3}{x^4}$, $\chi^{\text{IV}}(x) = \frac{2\,.\,3\,.\,4\,.\,5}{x^6}$, ..., we have

$$\frac{x}{r^3} = \chi(x) - \frac{1}{2}\frac{1}{2\,!}(y^2+z^2)\chi''(x) + \frac{1\,.\,3}{2\,.\,4}\frac{1}{4\,!}(y^2+z^2)^2\chi^{\text{IV}}(x) - \ldots , \tag{60}$$

which, being of the same form as (42), leads to no solution additional to those which might have been obtained from $1/r$.

In studying Bessel's functions, it is more natural to begin with solutions of (2) than with solutions of (1). We have as an illustration the physical problem of the vibration of a circular drumhead, or of wave-motion in a circular sheet of water, the periodicity of the phenomenon suggesting circular time-functions.

By lemma (viii) we have as a solution of (2)

$$\frac{1}{\sqrt{x^2-(y^2+z^2)}} = \frac{1}{x} + \frac{1}{2}\frac{y^2+z^2}{x^3} + \frac{1\,.\,3}{2\,.\,4}\frac{(y^2+z^2)^2}{x^5} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\frac{(y^2+z^2)^3}{x^7} + \ldots . \tag{61}$$

Putting $\chi(x) = 1/x$, this becomes

$$\chi(x) + \frac{1}{2}\frac{1}{2\,!}(y^2+z^2)\chi''(x) + \frac{1\,.\,3}{2\,.\,4}\frac{1}{4\,!}(y^2+z^2)^2\chi^{\text{IV}}(x) + \ldots . \tag{62}$$

Suppose, now, that x, y, z are replaced by t, x, y in this expression and in equation (2). Let $\psi(t) = \cos t$; then $\psi''(t) = -\cos t$, $\psi^{\text{IV}}(t) = \cos t$, ... , and we have as a solution of (2) thus altered

$$\cos t\left[1 - \frac{1}{2^2}(x^2+y^2) + \frac{1}{2^2\,4^2}(x^2+y^2)^2 - \ldots\right] = \cos t\, J_0(\varpi), \tag{63}$$

where $\varpi^2 = x^2 + y^2$.

If $\psi(x) = x^n$ we have $\sigma^n P_n\left(\frac{x}{\sigma}\right)$ as a solution of (2), σ^2 denoting $x^2 - (y^2+z^2)$. This solution could have been obtained from $r^n P_n\left(\frac{x}{r}\right)$ by lemma (iv).

By lemma (viii),

$$\frac{1}{t^2-(x^2+y^2+z^2)} \text{ or } \frac{1}{t^2-r^2} \tag{64}$$

is a solution of (2) when x, y, z, w are replaced by t, x, y, z. In

$$\frac{1}{t^2 - r^2} = \frac{1}{t^2} + \frac{r^2}{t^4} + \frac{r^4}{t^6} + \frac{r^6}{t^8} + \ldots. \tag{65}$$

put $\chi(t) = 1/t^2$ and it becomes

$$\frac{1}{t^2 - r^2} = \chi(t) + \frac{r^2}{3!}\chi''(t) + \frac{r^4}{5!}\chi^{\text{IV}}(t) + \ldots. \tag{66}$$

Let $\psi(t) = \sin t$; we then have as a solution of (2)

$$V_4 = \sin t \, \frac{\sin r}{r}. \tag{67}$$

The solutions obtained in the next section are also of the kind specified in lemma (v), as will be shown presently.

§ 5. *Solutions Obtained by an Application of Taylor's Theorem.*

Suppose $\phi(\alpha x + \beta y + \gamma z + \ldots)$ to be expanded in powers of $\beta y + \gamma z + \ldots$ thus,

$$\phi(\alpha x) + \frac{\beta y + \gamma z + \ldots}{1!\,\alpha}\phi'(\alpha x) + \frac{(\beta y + \gamma z + \ldots)^2}{2!\,\alpha^2}\phi''(\alpha x) + \frac{(\beta y + \gamma z + \ldots)^3}{3!\,\alpha^3}\phi'''(\alpha x) + \ldots, \tag{68}$$

the accent denoting differentiation with respect to x. (If the accent denote differentiation with respect to αx, then, of course, α would not appear in the denominator.)

Equation (1) will be satisfied by this expression if

$$\alpha^2 + \beta^2 + \gamma^2 + \ldots = 0; \tag{69}$$

and equation (2) if

$$\alpha^2 - (\beta^2 + \gamma^2 + \ldots) = 0. \tag{70}$$

For, differentiating the coefficient of $\phi^{(\nu)}(\alpha x)$ twice with respect to each of the variables $y, z, \ldots$ and adding the results together, we have

$$\frac{\beta^2(\beta y + \gamma z + \ldots)^{\nu-2}}{(\nu-2)!\,\alpha^\nu} + \frac{\gamma^2(\beta y + \gamma z + \ldots)^{\nu-2}}{(\nu-2)!\,\alpha^\nu} + \ldots.$$

The term

$$\frac{(\beta y + \gamma z + \ldots)^{\nu-2}}{(\nu-2)!\,\alpha^{\nu-2}}\phi^{(\nu-2)}(\alpha x)$$

twice differentiated with respect to x becomes

$$\frac{\alpha^2(\beta y + \gamma z + \ldots)^{\nu-2}}{(\nu-2)!\,\alpha^\nu}\phi^{(\nu)}(\alpha x).$$

Thus it is seen that if $\phi(\alpha x+\beta y+\gamma z+\ldots.)$ be substituted in (1) or (2) the coefficient of the resulting $\phi^{(\nu)}(\alpha x)$ will be zero, and so (1) or (2) will be satisfied when the relation (69) or (70) obtains. Or what amounts to the same thing, the given $(y, z, \ldots.)$-coefficient of $\phi^{(\nu-2)}(\alpha x)$, added to the $\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+\ldots.\right)$-derivative of $\phi^{(\nu)}(\alpha x)$, is zero, and so this comes under lemma (v).

[It may be noted here that (68) is a solution of the equations

$$\frac{\partial^n V}{\partial x^n}+\frac{\partial^n V}{\partial y^n}+\frac{\partial^n V}{\partial z^n}+\ldots. \quad =0, \tag{71}$$

$$\frac{\partial^n \mathrm{V}}{\partial x^n}-\left(\frac{\partial^n \mathrm{V}}{\partial y^n}+\frac{\partial^n \mathrm{V}}{\partial z^n}+\ldots.\right)=0, \tag{72}$$

provided that, in the respective equations,

$$\alpha^n+\beta^n+\gamma^n+\ldots. \quad =0, \tag{73}$$

$$\alpha^n-(\beta^n+\gamma^n+\ldots.)=0. \tag{74}$$

This can be shown by differentiating the coefficient of $\phi^{(\nu)}(\alpha x)$ n times with respect to $y, z, \ldots.$, and adding the results together, and by differentiating the term

$$\frac{(\beta y+\gamma z+\ldots.)^{\nu-n}}{(\nu-n)!\,\alpha^{\nu-n}}\,\phi^{(\nu-n)}(\alpha x)$$

n times with respect to x.]

The whole of the expression $\phi(\alpha x+\beta y+\gamma z+\ldots.)$ satisfies (1) or (2). Consequently, if it can be thrown into parts linearly independent of one another, each such part must be a solution.

For separating the solutions of (1) just referred to, it is convenient to make use of a complex variable $ax+iby+jcz+kdw+\ldots.$ of p dimensions, where the original units are

$$1,\quad i=\sqrt{-1},\quad j=\sqrt{-1},\quad k=\sqrt{-1},\quad \ldots., \tag{75}$$

and where

$$a^2=b^2+c^2+d^2+\ldots.. \tag{76}$$

Let p denote the dimension of the complex or the number of original units; then the total number of units in the $(1, i, j, \ldots.)$-algebra is

$$p+c_2(p-1)+c_3(p-1)+\ldots.+c_{p-1}(p-1). \tag{77}$$

Here $c_2(p-1)$ means the number of combinations of $p-1$ things taken two at a time.

Similarly, for separating the solutions of (2) it is convenient to make use of a complex variable $ax + \mathrm{i}by + \mathrm{j}cz + \mathrm{k}dw + \ldots.$ of p dimensions, where the original units are

$$1, \quad \mathrm{i} = \sqrt{+1}, \quad \mathrm{j} = \sqrt{+1}, \quad \mathrm{k} = \sqrt{+1}, \quad \ldots., \tag{78}$$

and where

$$a^2 = b^2 + c^2 + d^2 + \ldots..$$

The parts of (68) thus separated may in the first instance be denoted by the Italic symbols

$$R_1, \quad iR_i, \quad jR_j, \quad ijR_{ij}, \quad \ldots., \tag{79}$$

and in the second by the Roman symbols

$$\mathrm{R}_1, \quad \mathrm{i}\mathrm{R}_\mathrm{i}, \quad \mathrm{j}\mathrm{R}_\mathrm{j}, \quad \mathrm{ij}\mathrm{R}_\mathrm{ij}, \quad \ldots.. \tag{80}$$

If $p = 2$, the two solutions of (1) are the ordinary X, Y of the function-theory; the two solutions of equation (2) are somewhat analogous functions which may be denoted by Roman X, Y.

In the case $p = 3$, we have as solutions of (1)

$$R_1 + iR_i + jR_j + ijR_{ij} = \phi(ax + iby + jcz) = \phi\,[ax + i(by - ijcz)]\,; \tag{81}$$

$$\begin{aligned} R_1 &= \phi(ax) - \frac{b^2y^2 + c^2z^2}{2!\,a^2}\,\phi''(ax) \\ &\qquad + \frac{b^4y^4 + 6\,b^2c^2y^2z^2 + c^4z^4}{4!\,a^4}\,\phi^{\mathrm{IV}}(ax) - \ldots., \end{aligned} \tag{82}$$

$$\begin{aligned} R_i &= \frac{by}{a}\,\phi'(ax) - \frac{b^3y^3 + 3\,bc^2yz^2}{3!\,a^3}\,\phi'''(ax) \\ &\qquad + \frac{b^5y^5 + 10\,b^3c^2y^3z^2 + 5\,bc^4yz^4}{5!\,a^5}\,\phi^{\mathrm{V}}(ax) - \ldots., \end{aligned} \tag{83}$$

$$\begin{aligned} R_j &= \frac{cz}{a}\,\phi'(ax) - \frac{c^3z^3 + 3\,cb^2zy^2}{3!\,a^3}\,\phi'''(ax) \\ &\qquad + \frac{c^5z^5 + 10\,c^3b^2z^3y^2 + 5\,cb^4zy^4}{5!\,a^5}\,\phi^{\mathrm{V}}(ax) - \ldots., \end{aligned} \tag{84}$$

$$\begin{aligned} R_{ij} &= \frac{bcyz}{a^2}\,\phi''(ax) - \frac{b^3cy^3z + bc^3yz^3}{3!\,a^4}\,\phi^{\mathrm{IV}}(ax) \\ &\qquad + \frac{b^5cy^5z + 5\,b^3c^3y^3z^3 + bc^5yz^5}{5!\,a^6}\,\phi^{\mathrm{VI}}(ax) - \ldots.. \end{aligned} \tag{85}$$

$$\therefore\; R_1 - R_{ij} + \sqrt{-1}\,(R_i + R_j) = \sum_{\nu=0}^{\nu=n} (-1)^{\nu/2}\,\frac{(by + cz)^\nu}{\nu!\,a^\nu}\,\phi^{(\nu)}(ax). \tag{86}$$

In all these cases the accent denotes differentiations with respect to x.

For the corresponding solutions of equation (2) we have

$$\mathrm{R}_1 + \mathrm{i}\,\mathrm{R}_{\mathrm{i}} + \mathrm{j}\,\mathrm{R}_{\mathrm{j}} + \mathrm{i}\mathrm{j}\,\mathrm{R}_{\mathrm{ij}} = \phi\,(ax + \mathrm{i}\,by + \mathrm{j}\,cz) = \phi\,[(ax + \mathrm{i}\,by + \mathrm{ij}\,cz)]. \qquad (87)$$

The R's may be obtained from the preceding expressions by changing each minus sign into a plus.

If $\nu = \infty$,

$$R_1 + R_{ij} = \cos\left[\frac{by - cz}{a}\frac{\partial}{\partial x}\right]\phi\,(ax), \qquad (88)$$

$$R_1 - R_{ij} = \cos\left[\frac{by + cz}{a}\frac{\partial}{\partial x}\right]\phi\,(ax), \qquad (89)$$

$$R_i + R_j = \sin\left[\frac{by + cz}{a}\frac{\partial}{\partial x}\right]\phi\,(ax), \qquad (90)$$

$$R_i - R_j = \sin\left[\frac{by - cz}{a}\frac{\partial}{\partial x}\right]\phi\,(ax). \qquad (91)$$

For corresponding combinations of the R's, replace the symbols cos, sin by cosh, sinh.

As already stated, (68) is a solution of (1) or (2) of the form referred to in lemma (v), and so the linearly independent parts R_1, R_i, R_j, R_{ij}, ... are severally solutions of that form.

A lemma similar to (v) is applicable to solutions of (71) and (72); the 2's throughout equation (5) must then be changed to n's.

§ 6. *Applications of Complex Variables of Two or More Dimensions.*

Instances of the use of the ordinary two-dimensional variable $x + iy$, or $x - iy$, and the variable $\log r + i\theta$ $\left(\text{where } r^2 = x^2 + y^2,\ \theta = \tan^{-1}\frac{y}{x}\right)$ for finding or expressing solutions of equation (1) need not here be given, since they frequently occur in cartography, in works on hydrodynamics, electricity and magnetism, and other branches of mathematical physics.

A few examples of such solutions of (1) and (2) will now be given as may be expressed by aid of the other complex variables mentioned in the preceding section. The examples show that solutions of this kind, or aggregates of them, may have physical applications; also, that they are in many ways analogous to solutions made up of the ordinary X- and Y-functions. The conception of these complex quantities may not be of great importance in the obtaining of solutions satisfying prescribed conditions; but when solutions have been once obtained, this idea often facilitates their inspection.

Example (1). Any function ϕ of the variable $x + \mathrm{i}\, y$, where $\mathrm{i} = \sqrt{+1}$, gives two solutions of equation (2); that is

$$\phi\,(x + \mathrm{i}\, y) = \mathrm{R}_1 + \mathrm{i}\, \mathrm{R}_{\mathrm{i}} \tag{92}$$

gives R_1, R_{i} as solutions.

If for x we put $\varkappa\, t$, and for y, x, then R_1, R_{i} are solutions of

$$\frac{\partial^2 \mathrm{V}}{(\partial \varkappa t)^2} - \frac{\partial^2 \mathrm{V}}{\partial x^2} = 0, \quad \text{or} \quad \frac{\partial^2 \mathrm{V}}{dt^2} - \varkappa^2 \frac{\partial^2 \mathrm{V}}{\partial x^2} = 0. \tag{93}$$

For a vibrating string, the displacement V should be periodic with respect to t. ϕ may be put equal to $A_1 \cos \frac{\pi}{l}$ or $A_1 \sin \frac{\pi}{l}$, where l denotes the length of the cord. Suppose $\phi = A_1 \sin \frac{\pi}{l}$; then

$$\mathrm{R}_1 + \mathrm{i}\, \mathrm{R}_{\mathrm{i}} = \mathrm{A}_1 \sin \frac{\pi}{l}\,(\varkappa\, t + \mathrm{i}\, x), \tag{94}$$

and so

$$\mathrm{R}_1 = \mathrm{A}_1 \sin \frac{\pi \varkappa t}{l} \cos \frac{\pi x}{l}, \quad \mathrm{R}_{\mathrm{i}} = \mathrm{A}_1 \cos \frac{\pi \varkappa t}{l} \sin \frac{\pi x}{l}. \tag{95}$$

The displacement R_{i} becomes zero when $x = 0$, $x = l$; and $\frac{\partial \mathrm{R}_{\mathrm{i}}}{\partial t} = 0$ when $t = 0$. The problem of having the string initially at rest, and conforming to a prescribed curve whose equation is

$$y \text{ or } \mathrm{R}_{\mathrm{i}} = f(x),$$

generally requires for its solution an infinite number of terms like R_{i}. That is, when $t = 0$, y is of the form

$$y = \mathrm{A}_1 \sin \frac{\pi x}{l} + \mathrm{A}_2 \sin \frac{2 \pi x}{l} + \mathrm{A}_3 \sin \frac{3 \pi x}{l} + \ldots\ldots \tag{96}$$

In this case the function ϕ is really

$$\mathrm{R}_1 + \mathrm{i}\, \mathrm{R}_{\mathrm{i}} = \mathrm{A}_1 \sin \frac{\pi}{l}\,(\varkappa t + \mathrm{i}\, x) + \mathrm{A}_2 \sin \frac{2 \pi}{l}\,(\varkappa\, t + \mathrm{i}\, x) + \mathrm{A}_3 \sin \frac{3 \pi}{l}\,(\varkappa\, t + \mathrm{i}\, x) + \ldots\ldots \tag{97}$$

Example (2). Using the variable $a\, x + \mathrm{i}\, b\, y + \mathrm{j}\, c\, z$, where $\mathrm{i} = \sqrt{+1}$, $\mathrm{j} = \sqrt{+1}$ and, of course, $a^2 = b^2 + c^2$, any function

$$\phi\,(a\, x + \mathrm{i}\, b\, y + \mathrm{j}\, c\, z) = \mathrm{R}_1 + \mathrm{i}\, \mathrm{R}_{\mathrm{i}} + \mathrm{j}\, \mathrm{R}_{\mathrm{j}} + \mathrm{i}\mathrm{j}\, \mathrm{R}_{\mathrm{ij}} \tag{98}$$

gives four solutions of (2). If for x we put $\varkappa t$, for y, x, and for z, y, the R's are solutions of the equation

$$\frac{\partial^2 \mathrm{V}}{\partial t^2} - \varkappa^2 \left(\frac{\partial^2 \mathrm{V}}{\partial x^2} + \frac{\partial^2 \mathrm{V}}{\partial y^2} \right) = 0. \tag{99}$$

Giving to b, c, the values $\frac{m\pi}{h_1}$, $\frac{n\pi}{h_2}$, and so to a the value $\sqrt{\frac{m^2}{h_1^2}+\frac{n^2}{h_2^2}}\,\pi$, and making $\phi =$ cosine, we have

$$\left.\begin{aligned} R_{ij} &= -\cos \pi x t \sqrt{\frac{m^2}{h_1^2}+\frac{n^2}{h_2^2}} \sin\frac{m\pi x}{h_1} \sin\frac{n\pi y}{h_2}, \\ R_1 &= \cos \pi x t \sqrt{\frac{m^2}{h_1^2}+\frac{n^2}{h_2^2}} \cos\frac{m\pi x}{h_1} \cos\frac{n\pi y}{h_2}.^{*} \end{aligned}\right\} \quad (100)$$

Functions like these are suitable for representing the free vibrations of a rectangular or square drum-head, or for representing the motion of long stationary waves (*i. e.*, oscillations) in a rectangular or square tank of uniform depth. See example (12) below.

Example (3). Taking as variable $ax + iby + jcz$, where $i = \sqrt{-1}$, $j = \sqrt{-1}$, $a^2 = b^2 + c^2$, and letting

$$\phi(ax + iby + jcz) = e^{-(ax + iby + jcz)}, \qquad (101)$$

we have

$$R_{ij} = e^{-ax} \sin by \sin cz. \qquad (102)$$

If for b, c we write

$$\frac{(2m+1)}{h_1}\pi, \quad \frac{(2n+1)}{h_2}\pi,$$

and so for a,

$$\sqrt{\frac{(2m+1)^2}{h_1^2}+\frac{(2n+1)^2}{h_2^2}}\,\pi,$$

we have

$$R_{ij} = e^{-\pi x\sqrt{\frac{(2m+1)^2}{h_1^2}+\frac{(2n+1)^2}{h_2^2}}} \sin\frac{2m+1}{h_1}\pi y \sin\frac{2n+1}{h_2}\pi z. \qquad (103)$$

This solution of equation (1) is applicable to the potential at points within an infinitely long rectangular prism, where the potential at the sides is zero, and that at the base is a given constant.†

Example (4). Taking $x + iy \cos\beta + jz \sin\beta$ as the variable, and $e^{-i(\,)}$ as the function, we have

$$\phi(ax + iby + jcz) = e^{-i(x + iy\cos\beta + jz\sin\beta)}, \qquad (104)$$

which may be written

$$e^{y\cos\beta - ijz\sin\beta} \cos x - i \sin x;$$

$$\left.\begin{aligned} \therefore\ R_1 + ij R_{ij} &= e^{y\cos\beta - ijz\sin\beta}\cos x, \\ iR_i + jR_j &= -i e^{y\cos\beta - ijz\sin\beta}\sin x \end{aligned}\right. \qquad (105)$$

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[*] See Byerly, "Fourier's Series, etc.," pp. 126–134.

[†] Byerly, pp. 138, 139.

are solutions of equation (1). If we do not care to distinguish between the 1-terms and the ij-terms, or the i-terms and the j-terms, we may put $ij = \pm 1$ and $j = \pm i$ in the above results. Multiplying by suitable time-factors and combining, we obtain as a solution of (1)

$$e^{y \cos\beta - z \sin\beta} \cos(x - \varkappa t). \tag{106}$$

This denotes the velocity-potential for a train of "edge" waves advancing in the x-direction, the shore sloping at an angle β.

This solution is due to Stokes.*

Example (5). If $a^2 = 1$, $b^2 = \frac{1}{2}$, $c^2 = \frac{1}{2}$, and $\phi =$ cosine, we have

$$\phi(ax + iby + jcz) = \cos\left(x + \frac{i}{\sqrt{2}} y + \frac{j}{\sqrt{2}} z\right); \tag{107}$$

$$\therefore\ R_1 = \cos x \cosh\frac{y}{\sqrt{2}} \cosh\frac{z}{\sqrt{2}}. \tag{108}$$

This expression, when multiplied by the time-factor, denotes the velocity-potential of a standing wave in a V-shaped canal whose sides make angles of 45° with the vertical. This type was discovered by Kelland.†

$$-R_{ij} = \cos x \sinh\frac{y}{\sqrt{2}} \cosh\frac{z}{\sqrt{2}}$$

denotes the velocity-potential of an asymmetrical mode, noticed by Greenhill.‡

Example (6). Considering now two variables $x + ix$, $x + i\frac{\sqrt{3}}{2} y + j\frac{1}{2} z$, whose coefficients evidently satisfy the relation $a^2 = b^2 + c^2$, the R's of

$$A \cos(x + iz) + C \cos\left(x + i\frac{\sqrt{3}}{2} y + j\frac{1}{2} z\right) \tag{109}$$

and of

$$B \sin(x + iz) + D \sin\left(x + i\frac{\sqrt{3}}{2} y + j\frac{1}{2} z\right) \tag{110}$$

are evidently solutions of equation (1); so will be the R_1 of the former added to the R_i of the latter. The result is

$$\cos x \left[A \cosh z + B \sinh z + \cosh\frac{\sqrt{3}}{2} y \left(\cosh\frac{1}{2} z + D \sinh\frac{1}{2} z\right)\right]. \tag{111}$$

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[*] Lamb, "Hydrodynamics," 2d Ed. (1895), p. 431; 3d Ed. (1905), p. 424.

[†] *Ibid.*, 2d Ed., p. 432; 3d Ed., p. 424.

[‡] *Ibid.*, 3d Ed., p. 425.

This expression, when multiplied by the time-factor, is the velocity-potential for a type of standing waves in a canal whose sides are inclined to the vertical at an angle of 60°. A, B, C, D are determined by the boundary and surface conditions. This comprehensive type was discovered by Macdonald.*

Example (7). If

$$\phi(ax + iby + jcz) = \log(ax + iby + jcz), \tag{112}$$

then

$$R_i - R_j = \tan^{-1}\left(\frac{by + cz}{ax}\right) \tag{113}$$

must be a solution of equation (1), provided, of course, that $a^2 = b^2 + c^2$.

Example (8). If in

$$\phi(ax + iby + jcz) = (ax + iby + jcz)^2 \tag{114}$$

we make $a = \sqrt{2}$, $b = 1$, $c = 1$, then

$$\left.\begin{aligned} R_1 &= 2x^2 - y^2 - z^2, & R_i &= 2\sqrt{2}\,xy, \\ R_j &= 2\sqrt{2}\,xz, & R_{ij} &= 2yz. \end{aligned}\right\} \tag{115}$$

If we make $c = 0$, and so $a = 1$, $b = 1$, then

$$R_1 = x^2 - y^2, \qquad R_i = 2xy. \tag{116}$$

These are rational integral functions of degree 2 satisfying equation (1), and so are harmonic functions.

Replacing the exponent 2 by 3, 4, ..., rational functions of those degrees are obtained which satisfy equation (1). This fact has been noted by Laurent.† For the exponent 3 we have

$$\begin{aligned} R_1 + iR_i + jR_j + ijR_{ij} &= \sqrt{2}(2x^3 - 3xy^2 - 3xz^2) \\ &+ i(6x^2y - 3yz^2 - y^3) + j(6x^2z - 3y^2z - z^3) + ij\,6\sqrt{2}xyz. \end{aligned} \tag{117}$$

But

$$R_1 = \sqrt{2}\,[2x^3 - 3x(y^2 + z^2)], \tag{118}$$

having circular symmetry about the x-axis, is, like $2x^2 - y^2 - z^2$, a zonal harmonic.

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* See Lamb, "Hydrodynamics," 2d Ed., p. 434; 3d Ed., p. 426.

† "Il est bon d'observer que, si

$$\alpha^2 + \beta^2 + 1 = 0, \quad (\alpha x + \beta y + z)^m$$

satisfait à $\Delta_2 u = 0$ et $(\alpha \sin\theta \cos\psi + \beta \sin\theta \sin\psi + \cos\theta)^m$ est une fonction sphérique du degré m. Une somme de $2m + 1$ semblables fonctions sera la fonction sphérique la plus générale." H. Laurent, *Traité d'Analyse*, Tome VI, pp. 297, 298.

If we raise $\sqrt{2}x + iy + jz$ to the fourth power, the harmonic function R_1 thus obtained no longer possesses circular symmetry about the x-axis, although symmetrical in y^2 and z^2; for then

$$R_1 = 4x^4 - 12x^2y^2 - 12x^2z^2 + 6y^2z^2 + y^4 + z^4. \tag{119}$$

Suppose we raise the variable $y + iz$ to the fourth power; its R_1 or R_i must satisfy (1). [Cf. lemma (iii)].

Adding its R_1, $= -6y^2z^2 + y^4 + z^4$, to twice the preceding R_1, we have a zonal harmonic of the fourth order

$$8x^4 - 24x^2(y^2 + z^2) + 3(y^2 + z^2)^2. \tag{120}$$

Were it desirable to do so, it seems probable that the zonal harmonic of any order could be obtained by combining the R_1's of this power of variables of two and three dimensions.

Example (9). Still considering the case where $p = 3$, it will be noticed that the complex quantity $ax + iby + jcz$ contains one effective variable parameter. Let $a = 1$; then by (76), the quantity may be written $x + iby + j\sqrt{1 - b^2}z$. Hence, whenever in the expansion of $\phi(x + iby + j\sqrt{1 - b^2}z)$ it is possible to pick out the entire coefficient of any power of b, that coefficient is a solution of (1). This can be easily done for

$$(x + iby + j\sqrt{1 - b^2}z)^n, \tag{121}$$

where n is a positive integer. The solutions thus obtained are $2n + 1$ in number, as may be seen upon considering the four kinds (82)–(85), the first containing no odd powers of y or z:

$$\left.\begin{array}{lcc}
 & n \text{ odd} & n \text{ even} \\
\text{From } R_1, & 1 + \frac{n-1}{2} & 1 + \frac{n}{2} \\
\text{"} \quad R_i, & \frac{n+1}{2} & \frac{n}{2} \\
\text{"} \quad R_j, & 1 + \frac{n-1}{2} & 1 + \frac{n-2}{2} \\
\text{"} \quad R_{ij}, & \frac{n-1}{2} & \frac{n}{2} \\
\hline
\text{Total,} & 2n + 1 & 2n + 1
\end{array}\right\} \tag{122}$$

The coefficients of the various powers of b (including those of b^0) in the several kinds of terms may therefore be taken as the $2n + 1$ independent solutions of (1) of the n-th degree.

Now since the solution $r^n P_n$ is an homogeneous polynomial of degree n where y and z enter by even powers only, and which satisfies (1), it follows that it can be made up of linear combinations of the solutions of the first kind obtained by the expansion of (121). Hence, the surmise made at the end of example (8) turns out to be correct.

We can assume that if any solution V be a regular function of x, y, z within a finite domain surrounding x', y', z', it can be expanded in powers of X, Y, Z, where $x = x' + X$, etc., and the series will there be absolutely and uniformly convergent. The various powers of the variables x, y, z will give rise to homogeneous polynomials of all degrees (not exceeding n) in X, Y, Z. Each is a solution of (1) because, by hypothesis, the whole series is a solution and because of the homogeneous character of (1). From what has just been shown, it follows that for any such polynomial of degree n there are at most $2n + 1$ independent solutions. Hence, if the expansion be carried to the n-th powers of X, Y, Z, we obtain at most $\sum\limits_{\nu=0}^{\nu=n} 2\nu + 1$ or $(n+1)^2$ solutions.*

Example (10). Taken in connection with the foregoing section, it seems highly probable that $\frac{1}{r}$ can be expressed in a similar manner by using the R_1 of an infinite number of terms of the form

$$\frac{1}{ax + iby + jcz}. \qquad (123)$$

We have

$$\left.\begin{aligned} &\frac{1}{2}R_1\left[\frac{1}{2(x+iy)}+\frac{1}{2(x+jz)}+\frac{\sqrt{2}}{\sqrt{2}\,x+iy+jz}\right],\\ &=\frac{1}{x}\left[1-\frac{1}{2}\frac{y^2+z^2}{x^2}+\frac{1.3}{2.4}\frac{(y^2+z^2)^2}{x^4}+\frac{1.3.5}{2.4.6}\frac{(y^2+z^2)^3}{x^6}+\dots\right],\end{aligned}\right\} \qquad (124)$$

which agrees with the value of $1/r$ (41) to the number of terms here given.

It may be noted that a term of the form (123) becomes infinite only when the variable becomes zero or lies upon one of the two lines in the yz-plane whose equations are

$$x = 0, \quad by = \pm cz. \qquad (125)$$

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* Taken mainly from an important paper (*q. v.*) by Professor E. T. Whittaker, entitled "On the Partial Differential Equations of Mathematical Physics," *Mathematische Annalen*, Vol. LVII. He appears to have been the first to show that (121) gives the general homogeneous polynomial solution of (1).

For, (123) when rationalized in denominator gives

$$\frac{ax - iby - jcz}{a^2x^2 + b^2y^2 + c^2z^2 - 2ijbcyz}$$
$$= \frac{(ax - iby - jcz)(a^2x^2 + b^2y^2 + c^2z^2 + 2ijbcyz)}{(a^2x^2 + b^2y^2 + c^2z^2)^2 - 4b^2c^2y^2z^2}. \tag{126}$$

The denominator of this expression vanishes whenever

$$(ax)^2 + (by \mp cz)^2 = 0.$$

Example (11). Consider the R_1 terms of (123) or (126). They are

$$\frac{ax(a^2x^2 + b^2y^2 + c^2z^2)}{(a^2x^2 + b^2y^2 + c^2z^2)^2 - 4b^2c^2y^2z^2}. \tag{127}$$

Putting $a = 1$ and eliminating c by means of the equation $-a^2 + b^2 + c^2 = 0$, the fraction becomes

$$\frac{x[x^2 + z^2 + b^2(y^2 - z^2)]}{(x^2 + z^2)^2 + 2b^2(x^2y^2 - x^2z^2 - y^2z^2 - z^4) + b^4(y^2 + z^2)^2} \tag{128}$$

and may, for brevity, be written

$$\frac{x[x^2 + z^2 + (y^2 - z^2)b^2]}{p + qb^2 + rb^4}$$
$$= x[x^2 + z^2 + (y^2 - z^2)b^2]\left[\frac{1}{p} - \frac{q}{p^2}b^2 + \frac{1}{p}\left(\frac{-r}{p} + \frac{q^2}{p^2}\right)b^4 - \dots\right], \tag{129}$$

a development in descending powers of $(x^2 + z^2)^2$.

The coefficient of b^0 is

$$\frac{x}{x^2 + z^2}; \tag{130}$$

the coefficient of b^2 is

$$x\frac{z^2 - y^2}{(x^2 + z^2)^2} + 4x\frac{y^2z^2}{(x^2 + z^2)^3}; \tag{131}$$

either is a solution of equation (1).

Example (12). For phenomena periodic in time and involving the differential relation

$$\frac{\partial^2}{\partial t^2} = \varkappa^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \tag{132}$$

ϕ should consist of a number of sine or cosine terms or of both. Here suppose ϕ to consist of terms of the form

$$\cos(\varkappa at + \mathrm{i}bx + \mathrm{j}cy). \tag{133}$$

If the phenomena be *simply* harmonic in time, then the square of the a of each term of the function will be the same. The R's become

$$\left.\begin{aligned} R_1 &= \cos \varkappa a t \cos b x \cos c y + \cos \varkappa a t \cos b' x \cos c' y + \ldots, \\ -R_i &= \sin \varkappa a t \sin b x \cos c y + \sin \varkappa a t \sin b' x \cos c' y + \ldots, \\ -R_j &= \sin \varkappa a t \cos b x \sin c y + \sin \varkappa a t \cos b' x \sin c' y + \ldots, \\ -R_{ij} &= \cos \varkappa a t \sin b x \sin c y + \cos \varkappa a t \sin b' x \sin c' y + \ldots, \end{aligned}\right\} \quad (134)$$

where $a^2 = b^2 + c^2 = b'^2 + c'^2 = \ldots\ldots$

As already noted, functions of the form $\cos \varkappa a t \sin b x \sin c y$ or $\cos \varkappa a t \cos b x \cos c y$ are suitable for representing the vibration of a rectangular drumhead or the oscillation of a shallow rectangular body of water. It remains to suggest here that by taking a sufficient number of similar terms it would probably be possible to thereby represent these phenomena in triangular, circular, or any other shaped areas. For instance, the solution given by Lamé* for the vibration of a membrane whose form is that of an equilateral triangle will be seen to consist of terms of the form (134) as soon as his trilinear coördinates are replaced by Cartesian coördinates conveniently chosen. We shall here confine ourselves to the oscillations of shallow bodies of water.

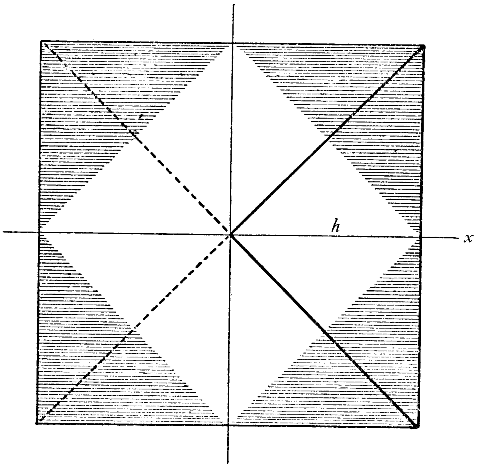

FIG. 1.

One mode of oscillation for a 45° triangle is shown in Fig. 1. The vertical displacement, ζ, is proportional to

$$\cos \frac{\pi \varkappa t}{h} \left(\cos \frac{\pi x}{h} + \cos \frac{\pi y}{h} \right), \quad (135)$$

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* "Leçons sur la théorie mathématique de l'élasticité des corps solides," p. 132.

which is of the form

$$\cos \varkappa a t \left[\cos b x \cos c y + \cos b' x \cos c' y\right],$$

where $c^2 = b'^2 = 0$, $b^2 = c'^2 = a^2$, and $\varkappa^2 = g \times$ depth of water.

Had the coördinate axes been taken along the legs of the triangle, the above displacement would have been written as one term, viz.,

$$2 \cos \frac{\pi \varkappa t}{h} \cos \frac{\pi x}{\sqrt{2}\,h} \cos \frac{\pi y}{\sqrt{2}\,h}, \tag{136}$$

which is of the form

$$2 \cos \varkappa a t \cos b x \cos c y,$$

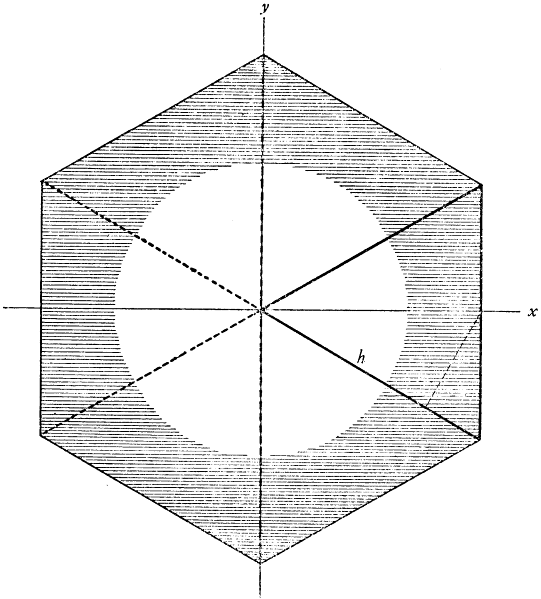

FIG. 2.

where $b^2 = c^2 = \frac{1}{2} a^2$. (Cf. Example 5, above.)

If the above expressions for vertical displacement be expanded in powers of x and y, the results are the same to two terms, viz.,

$$2 \cos \varkappa a t \left(1 - a^2 \frac{x^2 + y^2}{2^2}\right). \tag{137}$$

Considering now a square composed of four triangles, arranged as in the figure, we see that for this mode of oscillation there is a single nodal line consisting of four straight lines in the form of a square. Upon differentiating normally to any boundary of any of these triangles, or to the lines bisecting their right angles, it is easily seen that $\frac{\partial \zeta}{\partial \nu} = 0$.

One mode of oscillation for an equilateral triangle is shown in Fig. 2. Here ζ is proportional to

$$\cos\frac{\pi\varkappa t}{h}\left[2\cos\frac{\sqrt{3}\,\pi x}{2h}\cos\frac{\pi y}{2h}+\cos\frac{\pi y}{h}\right],^{*} \qquad (138)$$

which is of the form

$$\cos\varkappa a t\,[2\cos bx\cos cy+\cos b'x\cos c'y],$$

where $b^2=\frac{3}{4}a^2$, $c^2=\frac{1}{4}a^2$, $b'^2=0$, $c'^2=a^2$. (Cf. Example 6, above.)

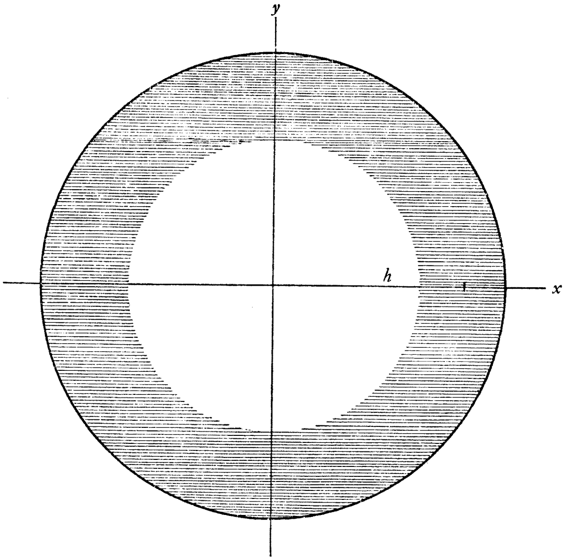

FIG. 3.

This, if expanded, gives for the first three terms

$$3\cos\varkappa a t\left[1-a^2\frac{x^2+y^2}{2^2}+a^4\frac{(x^2+y^2)^2}{2^2\,4^2}\right]. \qquad (139)$$

Considering now a hexagon composed of equilateral triangles as in the figure, we see that for this mode of oscillation there is a single nodal line resembling the circumference of a circle much more nearly than did the nodal line in the preceding case. Upon differentiating normally to any boundary of any of these triangles or to lines bisecting their angles at the center of the hexagon, it is easily seen that $\frac{\partial\zeta}{\partial\nu}=0$.

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* Coast and Geodetic Survey Report for 1900, p. 589; also Poincaré, "Leçons de Mécanique Céleste," Tome III, p. 371.

The vertical displacement for the slowest symmetrical mode for a circle is proportional to

$$\cos \kappa a t \, J_0 (a \varpi),$$

or

$$\cos \kappa a t \left[1 - a^2 \frac{x^2 + y^2}{2^2} + a^4 \frac{(x^2 + y^2)^2}{2^2\,4^2} - a^6 \frac{(x^2 + y^2)^3}{2^2\,4^2\,6^2} + \ldots . \right]. \quad (140)$$

That this satisfies the differential relation (132), has been shown under the heading "Application of Lemma (v)".

At the circumference where $\varpi = \varpi_1$, $\partial J (a \varpi) / \partial \varpi$ must be zero. Such a value of $a \varpi$ is $a \varpi_1 = 1.22\,\pi$, or $a = 1.22\,\pi / \varpi_1$; that is, ϖ_1 being given, a becomes fixed, and conversely. If $a \varpi = 0.7655\,\pi$, $J_0 (a \varpi)$ becomes zero and so does the displacement ζ.

$$\therefore\; 1.22 \frac{\varpi}{\varpi_1} = 0.7655; \text{ and so } \varpi = 0.628\,\varpi_1$$

is the radius of the nodal circle. Since the motion is symmetrical about the center, the vectorial angle ϕ is not involved; consequently, $\dfrac{\partial \zeta}{\partial \phi} = 0$ everywhere.

Here the "speed" (*i. e.*, $2\pi \div$ period) is κa or $1.22\,\kappa \pi / \varpi_1$, while the "speed" for a rectangular trough whose length is ϖ_1 is $\kappa \pi / \varpi_1$, κ denoting, as usual, the velocity of advance dependent upon the depth. The period for the circle is, therefore, $1/1.22 = 0.820$ times the period for a rectangular trough ϖ_1 in length.

Certain ratios for square, hexagonal and circular areas are given here:

	Height (ζ) at center $\div$ Average height along perimeter	Area enclosed by nodal line $\div$ Area of polygon	Virtual length (h) $\div$ Apothem of polygon
Square (Fig. 1)	− 2.000	0.500	1.000
Hexagon (Fig. 2)	− 2.418	0.399	0.866
Circle (Fig. 3)	− 2.483	0.394	0.820

Comparing expressions (135), (139) and (140), we are led to believe that the R_1 of a sufficient number of terms of the form $\cos (\kappa a t + \mathrm{i}\, b x + \mathrm{j}\, c y)$ could be made to represent this oscillation of a circular sheet of water, just as in the preceding example we were led to believe that the solution, $1/r$, of Laplace's equation could be regarded as the R_1 of an infinite number of terms of the form

$$\frac{1}{a x + i b y + j c z}.$$

Moreover, it has been shown that any zonal harmonic of order n can be regarded as the R_1 of a suitable number of terms of the form

$$(ax + iby + jcz)^n.$$

§ 7. *Differential Relations Connecting the R-Functions.*

If
$$\frac{d(R_1 + iR_i + jR_j + ijR_{ij})}{d(ax + iby + jcz)} \tag{141}$$

has a unique value at a given point, it must be free from the variable quantities dx, dy, dz. Whenever the numerator contains the denominator as a factor, we have the relations

$$\left.\begin{aligned} &\frac{1}{a}\left(\frac{\partial R_1}{\partial x} + i\frac{\partial R_i}{\partial x} + j\frac{\partial R_j}{\partial x} + ij\frac{\partial R_{ij}}{\partial x}\right) \\ &\qquad = \frac{1}{ib}\left(\frac{\partial R_1}{\partial y} + i\frac{\partial R_i}{\partial y} \underline{+j}\frac{\partial R_j}{\partial y} + ij\frac{\partial R_{ij}}{\partial y}\right) \\ &\qquad = \frac{1}{jc}\left(\frac{\partial R_1}{\partial z} + i\frac{\partial R_i}{\partial z} + j\frac{\partial R_j}{\partial z} + ij\frac{\partial R_{ij}}{\partial z}\right). \end{aligned}\right\} \tag{142}$$

$$\left.\begin{aligned} \therefore\ \frac{\partial R_1}{a\partial x} &= \frac{\partial R_i}{b\partial y} = \frac{\partial R_j}{c\partial z}, \\ \frac{\partial R_1}{b\partial y} &= -\frac{\partial R_i}{a\partial x} = -\frac{\partial R_{ij}}{c\partial z}, \\ \frac{\partial R_1}{c\partial z} &= -\frac{\partial R_j}{a\partial x} = -\frac{\partial R_{ij}}{b\partial y}, \\ -\frac{\partial R_1}{c\partial z} &= -\frac{\partial R_j}{b\partial y} = \frac{\partial R_{ij}}{a\partial x} \end{aligned}\right\} \tag{143}$$

are therefore the nine conditions for a monogenic function of the three-dimensional variable $ax + iby + jcz$.

When the Roman R's and i, j are used, all signs are positive in these relations.

If the variable have more than three dimensions, the additional relations can be readily determined.

For two-demensional variables we have, of course,

$$\left.\begin{aligned} &\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y}, \quad \frac{\partial X}{\partial y} = -\frac{\partial Y}{\partial x}, \\ &\frac{\partial \mathrm{X}}{\partial x} = \frac{\partial \mathrm{Y}}{\partial y}, \quad \frac{\partial \mathrm{X}}{\partial y} = \frac{\partial \mathrm{Y}}{\partial x}. \end{aligned}\right\} \tag{144}$$